Stabilization of the Inverted Pendulum by means of a nested saturation function

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Abstract. In this letter we present a technique to stabilize the inverted pendulum mounted on a cart. The pendulum is brought to its top position with zero displacement of the cart by using a nested saturation function. This can be done because the original system can be expressed as a chain of integrator with an additional nonlinear perturbation. Under the assumption that the pendulum angle is initialized above the upper-half plane, the obtained closed-loop system is semi-global asymptotically and locally exponentially stable.

1 Introduction

The stabilization of the inverted pendulum on a cart (**IPC**) is a very interesting problems. This device consists of a free vertical rotating pendulum with a pivot point mounted on a cart. The cart can be moved horizontally by means of an horizontal force, which is the control of the system. Because the pendulum angular acceleration can not be directly controlled, this system is a classic example of an under-actuated mechanical system, that is, it has fewer actuators than degrees-of-freedom. For this reason the majority of fully-actuated systems control techniques cannot be directly applied to stabilize this kind of systems. Maneuvers such as the stabilization around the unstable vertical position and others related to the stabilization around its homoclinic orbits are almost impossible to achieve (see[14],[9],[16]) because the **IPC** is not input-output linearizable using a static feedback [7]. Also, when the pendulum moves through the horizontal plane it looses controllability and other geometric properties [9]. On the other hand, a linearized model of the **IPC** is is locally controllable around the unstable equilibrium point and can be locally stabilized by a direct pole placement procedure [15].

There are two important problems related to the stabilization of this device. The first is swinging the pendulum up from the hanging position to the upright vertical position. An energy control strategy is usually applied for this purpose. Once the system is close to the desired top position with low enough speed (the inverted pendulum remains swinging while getting closer and closer to the origin), and suddenly, by means of a simple change in the controller, from the non-linear to the linear controller, it is possible to keep the pendulum in the

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desired equilibrium [3],[6],[9],[16],[14]. The second problem in importance consists in stabilization of the **IPC** around its unstable equilibrium point (which is defined when the angle position and the displacement of the cart are zero), assuming that the pendulum is initially above the horizontal plane, or lies inside an open vicinity of zero. In general, this vicinity defines a stability domain for the closed-loop system. Exist many works related with the second problem but a detailed review of the state of the art of the problem here treated is beyond the scope of this work. However, we refer the interested reader to the following references: [11],[19],[1],[2],[13],[12], [5].

In this paper we develop a simple strategy for the stabilization to the **IPC**. We transform the original system into a four-order integrator chain plus an additional nonlinear perturbation based in the procedure presented in [13]. Then, by applying the simple Lyapunov method, a controller based on nested saturated functions is introduced. Next, we show that the closed-loop solution is bounded, which allows to prove that the system is locally exponentially stable. The stability analysis of the whole four order system is fairly simple, as opposite to [13],[10], because, we neither use a fixed-point equation, nor a highly complex domain of attraction estimation. Also, we do not use the contraction mapping theorem to verify the convergence of all states to zero.

The remaining of this paper is organized as follows. Section 2 presents the dynamical model of the **IPC** and how this system is converted into an integrators chain, by means of some suitable transformations. In section 3 we present a stabilizing nonlinear controller for the **IPC**. The corresponding stability and convergence analysis is carried out in the same section. Section 4 presents some computer simulations and the conclusions are given in section 5.

2 Nonlinear Model

Consider the traditional **IPC**, as shown in Figure 1. This system is described by the set of normalized differential equations [12]:

$$\begin{aligned} &\cos\theta\ddot{x} + \ddot{\theta} - \sin\theta = 0, \\ &(1+\delta)\ddot{x} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta = u, \end{aligned} \tag{1}$$

where x is the normalized displacement of the cart, θ is the angle that the pendulum forms with the vertical, u is the horizontal normalized force applied to the cart, and $\delta > 0$ is a constant depending directly on the cart mass and the pendulum mass, respectively. Defining $v = \ddot{\theta}$ and canceling $\ddot{x}$ from the last two differential equations, we have after substituting:

$$u = (1+\delta)\tan\theta_1 - \theta_2^2\sin\theta_1 + v(\cos\theta_1 - \frac{1+\delta}{\cos\theta_1}),$$

into system (1) the following

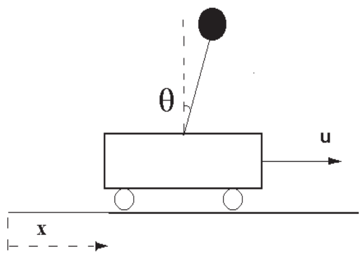

Fig. 1. The inverted pendulum cart system

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \tan\theta_1 - \tfrac{v}{\cos\theta_1}, \\
\dot{\theta}_1 &= \theta_2, \\
\dot{\theta}_2 &= v.
\end{aligned} \tag{2}$$

Where v is a fictitious controller acting in the coordinate θ_1. Of course, the above representation is validated for all $\theta_1 \in (-\pi/2, \pi/2)$. *Hereafter, we refer to this restriction as assumption* ***A1*** *and system (2) as a partial linearized model of the* ***IPC****.*

The main objective is to control the partial linearized system (2) under assumption **A1**. In other words we want to bring, both, the pendulum angle position and the cart displacement to zero.

2.1 Transforming the partial linear model into a chain of integrators:

From [13] we introduce

$$z_1 = g(\theta_1) + x_1 \; z_2 = g'(\theta_1)\theta_2 + x_2 \tag{3}$$

where the function g is selected such that the derivative of the variable z_2 does not depend directly on the control v, that is,

$$\dot{z}_2 = \tan\theta_1 + v(g'(\theta_1) - \frac{1}{\cos\theta_1}) + \theta_2^2 g''(\theta_1). \tag{4}$$

Consequently,

$$g'(\theta_1) = \frac{1}{\cos\theta_1}; \quad g(\theta_1) = \log\left(\frac{1 + \tan(\theta_1/2)}{1 - \tan(\theta_1/2)}\right), \tag{5}$$

these relations are well defined for $|\theta_1| < \pi/2$.

Now, from (3) and (5), we can write system (2), as follows,

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= \tan(\theta_1)(1 + \tfrac{\theta_2^2}{\cos\theta_1}), \\ \dot{\theta}_1 &= \theta_2, \\ \dot{\theta}_2 &= v. \end{aligned} \tag{6}$$

In order to express the above system as an integrators chain plus a nonlinear perturbation, the following global nonlinear transformations is introduced

$$\begin{aligned} w_1 &= \tan\theta_1, \quad w_2 = \sec^2\theta_1\theta_2, \\ v_f &= \sec^2\theta_1 v + 2\theta_2^2 \tan\theta_1 \sec^2\theta_1 \end{aligned} \tag{7}$$

which leads to

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= w_1 + \tfrac{w_1 w_2^2}{(1+w_1^2)^{3/2}}, \\ \dot{w}_1 &= w_2, \\ \dot{w}_2 &= v_f. \end{aligned} \tag{8}$$

2.2 Comment

A similar representation of model (8) was proposed in [10]. There, the control and the non-actuated coordinate are not completely uncoupled.That is the control acts directly on the additional nonlinear perturbation, and as consequence, the resulting closed-loop system has a more restrictive domain of attraction. In our case, the control action is completely uncoupled, so that, it is possible to increase the stability domain for all the initial conditions that belongs in the upper half plane.

3 Control strategy

A nested saturation function is suggested to use to control a nonlinear system that can be expressed, approximately, as a chain of integrators with a nonlinear perturbation. This technique, introduced in [18], has been used for the stabilization of a linear integrators chain and controlling mini-flying machines [20]. Thus, our stability problem will be solved as follows. First, a linear transformation is used to directly propose a stabilizing controller; then, it is shown that the proposed controller guarantees the boundedness of all states and, after a finite time, it is possible to ensure that all states converge to zero.
Definition: $\sigma_m(s) : R \to R$ is a linear saturation function, if it satisfies

$$\sigma_m(s) = \begin{cases} s & if \ |s| \leq m \\ m \ sign(s) & if \ |s| > m \end{cases}. \tag{9}$$

3.1 A feedback controller

Let us introduce the following linear transformations:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 1\,3\,3\,1 \\ 0\,1\,2\,1 \\ 0\,0\,1\,1 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix}, \tag{10}$$

then system (8) is transformed as,

$$\begin{aligned} \dot{q}_1 &= v_f + q_2 + q_3 + q_4 + 3\delta_a(q) \\ \dot{q}_2 &= v_f + q_3 + q_4 + \delta_a(q) \\ \dot{q}_3 &= v_f + q_4 \\ \dot{q}_4 &= v_f \end{aligned} \tag{11}$$

where the perturbation δ_a is given by

$$\delta_a(q) = q_4^2 G(q_3 - q_4), \tag{12}$$

and

$$G(w) = \frac{w}{(1+w^2)^{3/2}}, \tag{13}$$

for simplicity, we stand for $q = [q_1, q_2, q_3, q_4]$.
Remark 1: *Note that* $\max |G(w)| \leq k_0 = \frac{2}{3^{3/2}}$, *and it is achieved when* $w = 1/\sqrt{2}$.

Finally, a stabilizing controller may be readily proposed as:

$$v_f = -q_4 - k\sigma_\alpha\left(\frac{q_3 + \sigma_\beta(q_2 + \sigma_\gamma(q_1))}{k}\right). \tag{14}$$

where k is positive constant.

3.2 Boundedness of all states

We show in four steps that the proposed controller (14) ensures that all the states are bounded; moreover, the bound of each state depends directly on the controller parameters.[1]

First step. Define the positive definite function $V_4 = q_4^2/2$. Then the time derivative of V_4 is given by,

$$\dot{V}_4 = -q_4^2 - kq_4\sigma_\alpha(q_3/k + \sigma_\beta(q_2 + \sigma_\gamma(q_1))/k). \tag{15}$$

[1] Note that $|q_4(t)| \leq q_4(0)e^{-t} + \alpha$ and $|G(q_3 - q_4)| \leq k_0$. Therefore, the right hand of the closed loop system (14) and (11) is locally Lipschitz. Consequently, the states $\{q_1, q_2, q_3\}$ cannot have a finite time scape [8].

It is clear that $\dot{V}_4 < 0$, when $|q_4| \geq \alpha k$. Consequently, there is a finite time $T_1 > 0$ such that

$$|q_4(t)| < \alpha k \quad \forall t > T_1. \tag{16}$$

Second step. Let us analyze the behavior of the state q_3. Consider the definite positive function $V_3 = q_3^2/2$. Differentiating V_3, we obtain after substituting (14) into the third differential equation of (11):

$$\dot{V}_3 = -q_3 k \sigma_\alpha (q_3/k + \sigma_\beta(q_2 + \sigma_\gamma(q_1))/k), \tag{17}$$

where α and β are selected such that $\alpha > 2\beta$. Clearly, if $q_3 > \beta$, then $\dot{V}_3 < 0$ and there is a finite time $T_2 > T_1$ after which

$$|q_3(t)| < \beta \;\; forallt > T_2. \tag{18}$$

When the above condition is satisfied, the control v_f turns out to be

$$v_f = -q_4 - q_3 - \sigma_\beta(q_2 + \sigma_\gamma(q_1)) \; \forall t > T_2. \tag{19}$$

Third step. Substituting (19) into the second equation of (11), we obtain

$$\dot{q}_2 = -\sigma_\beta(q_2 + \sigma_\gamma(q_1)) + \delta_a(q). \tag{20}$$

Define a positive definite function $V_2 = q_2^2/2$. Differentiating V_2 along of the trajectories of (20) yields[2]

$$\dot{V}_2 = -q_2 \left(\sigma_\beta(q_2 + \sigma_\gamma(q_1)) + \delta_a(q)\right) \tag{21}$$

where β and γ must satisfy $\beta > 2\gamma + k_0\alpha^2 k^2$. Obviously, if $|q_2| > \gamma + k_0\alpha^2 k^2$ then $\dot{V}_2 < 0$. Hence, there exist a finite time $T_3 > T_2$ after which

$$|q_2| < \gamma + k_0 k^2 \alpha^2, \;\; \forall t > T_3. \tag{22}$$

Consequently, q_2 is bounded and the control v_f becomes

$$v_f = -q_4 - q_3 - q_2 - \sigma_\gamma(q_1), \;\; \forall t > T_3. \tag{23}$$

Fourth step. Substituting (23) into the first equation of (11), we obtain

$$\dot{q}_1 = -\sigma_\gamma(q_1) - 3\delta_a(q). \tag{24}$$

Now, define a positive definite function $V_1 = q_1^2/2$. By differentiating V_1 along of the trajectories of (24), we obtain

$$\dot{V}_1 = -q_1(\sigma_\gamma(q_1) + 3\delta_a(q)), \tag{25}$$

where parameter γ must be chosen such that $\gamma > 3k_0\alpha^2 k^2$. If $q_1 > 3k_0\alpha^2 k^2$, then $\dot{V}_1 < 0$, thus there exits a finite time $T_4 > T_3$ afterwards

$$|q_1| < 3k_0\alpha^2 k^2, \;\; \forall t > T_4. \tag{26}$$

[2] Recalling that after $t > T_3$, it has $|\delta_a(q)| \leq k_0\alpha^2 k^2$.

Consequently q_1 is also bounded. So, all the previous constraints on parameters α, β and γ can be summarized as

$$\alpha > 2\beta,\, \beta > 2\gamma + k_0 k^2 \alpha^2,\, \gamma > 3k_0 k^2 \alpha^2. \tag{27}$$

Manipulating the last inequalities, we have that

$$\alpha < 1/(14 k_0 k^2). \tag{28}$$

Thus, parameter k may be taken as $14k_0k^2 = 1$ and the set of control parameters may be selected as

$$\alpha = r,\, \beta = r/2,\, \gamma = 3r/14, \tag{29}$$

for all $0 < r \leq 1$.

3.3 Convergence of all states to zero

We will prove that the closed-loop system given by (11) and (14) is asymptotically stable and locally exponentially stable, provided that the controller parameter satisfies (27).

Note that after $t > T_4$, the control law is no longer saturated, that is,

$$v_f = -q_1 - q_2 - q_3 - q_4,$$

and the closed-loop system turns out to be, as

$$\begin{aligned} \dot{q}_1 &= -q_1 + 3\delta_a(q), \\ \dot{q}_2 &= -q_1 - q_2 + \delta_a(q), \\ \dot{q}_3 &= -q_1 - q_2 - q_3, \\ \dot{q}_4 &= -q_1 - q_2 - q_3 - q_4, \end{aligned} \tag{30}$$

with δ defined in (12) . Let us define the following Lyapunov function

$$V = \frac{1}{2} q^T q, \tag{31}$$

Now, differentiating V along the trajectories of (30), we obtain

$$\dot{V} = -q^T M q + (3q_1 + q_2)\delta_a(q) \tag{32}$$

where

$$M = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}. \tag{33}$$

being M positive definite with $\lambda_{\min}\{M\} = 1/2$.

From **Remark 1** and ((12), we easily have that the second term of the right hand of(32) satisfies

$$\begin{aligned} |(3q_1+q_2)\delta(q)| &< \tfrac{k_0}{2}\left|(3q_1+q_2)q_4^2\right|; \\ &< \tfrac{k_0}{2}(q_4^4+(3q_1+q_2)^2). \end{aligned} \tag{34}$$

So, $\dot{V}$ fulfills

$$\dot{V} < -\frac{1}{2}\left[q_1^2+q_2^2-k_0(3q_1+q_2)^2\right]-\frac{q_4^2}{2}(1-k_0q_4^2)-\frac{1}{2}q_3^2. \tag{35}$$

From definition of k_0 and recalling that $14k_0k^2=1$, we obtain that the previous inequality is strictly negative definite, since

$$q_1^2+q_2^2-k_0(3q_1+q_2)^2>0, \tag{36}$$

and

$$-1+k_0q_4^2 \leq -1+4k_0k^2<0. \tag{37}$$

Therefore, $\dot{V}$ is strictly negative definite, and the vector state q locally exponentially converges to zero, after $t>T_4$.

It should be noticed that, proceeding as described, we obtain that the system (11) in closed-loop with the controller (14) is globally asymptotically stable and locally exponentially stable, when the parameters satisfy the restriction (27). However, we can only assure converge to zero of the original states $(x,\theta,\dot{x},\dot{\theta})$, assuming that the initial angle of the pendulum belongs to the upper half plane, because (2) and (7) are well defined for $\theta\in(-\pi/2,\pi/2)$. That is, assumption **A1** is necessary to avoid the singular points $\theta=\pm\pi/2$.
From the above discussion, we have:

Proposition 1. Consider the partial linearization model of the IPC as described by (2), under assumption **A1**, in closed-loop with the controller:

$$v=-\theta_2\cos^2\theta_1-k\sigma_\alpha\left(\tfrac{q_3+\sigma_\beta(q_2+\sigma_\gamma(q_1))}{k}\right)\cos^2\theta_1-2\theta_2^2\tan\theta_1^2, \tag{38}$$

where $k=\sqrt{1/(14\times 2^{3/2})}$, q_1, q_2 and q_3 are given by

$$\begin{aligned} q_1 &= z_1+3z_2+3w_1+w_2;\ q_2=z_2+2w_1+w_2; \\ q_3 &= w_1+w_2, \end{aligned} \tag{39}$$

with

$$\begin{aligned} w_1 &= \tan\theta_1; & w_2 &= \theta_2\sec^2\theta_1; \\ z_1 &= \log\left(\tfrac{1+\tan(\theta_1/2)}{1-\tan(\theta_1/2)}\right)+x_1; & z_2 &= \theta_2/\cos\theta_1+x_2. \end{aligned} \tag{40}$$

Then the closed-loop system is semi-globally stable and locally exponentially stable provided that the control parameters α, β and γ satisfy the inequalities (27).

4 Numerical Simulations

The efficiency of the proposed control strategy was tested by computer simulations. The experiments were implemented in Matlab program. The controller parameter values were set as $\alpha = 0.99$, $\beta = 0.49$ and $\gamma = 0.214$, and the initial conditions were set as $\theta_1(0) = 1.18$ [rad], $\theta_2(0) = -0.05$ [rad/sec], $x_1(0) = -0.6$ and $x_2(0) = 0.5$.
Figure 2 and Figure 3 show the closed loop responses to the proposed controller (38), when it applied to the partial linearized model (2). As can be seen, the state x_1 converges very slowly to zero, in comparison with the state θ_1. This is because the cart position increases until the angle position of the pendulum approaches to zero. This event is expected since, firstly, the controller brings the pendulum into a small vicinity of zero, while, the cart position reaches its maximum, and secondly, the controller forces to move the cart slowly to the origin. Besides, it should be noticed that the control strategy was carried out with slowly movements. Finally, figure 4 shows the behavior of the control input v and the proposed energy function V, respectively. As can be seen, the control input v goes to zero and the Lyapunov function is decreasing after $t > 10$ and also converges to zero.

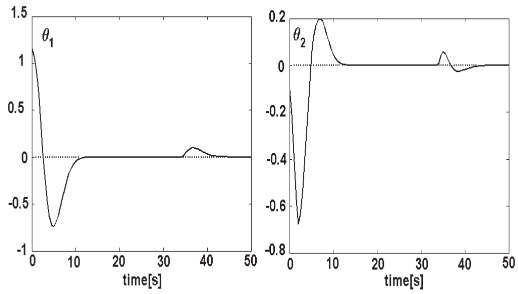

Fig. 2. Closed-loop response of the angle and the angular velocity to the proposed controller

5 Conclusions

A nested saturation-based controller for the stabilization of the **IPC** is presented, under assumptions that the initial value of the pendulum angle lies in the above horizontal plane. The fact that the **IPC** can be written (approximately) as a four cascaded integrators. Permits to use a nested saturation functions to design a stabilizing controller. The proposed controller makes the system be stable (under some restriction on the control parameters), and after some finite time assures that all states converge exponentially to zero. Physically, the control

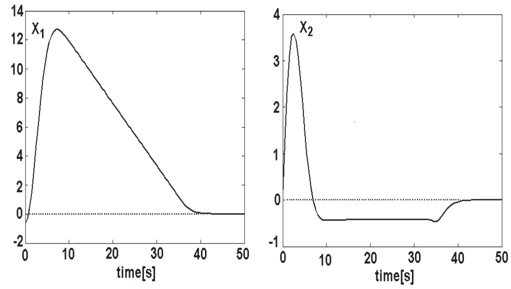

Fig. 3. Closed-loop response of the cart position and the cart velocity to the propose controller

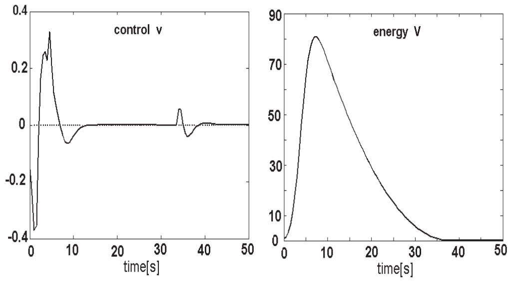

Fig. 4. Depict the behavior of the controller v and energy function V, respectively

strategy consist in bringing the pendulum close to the upper position, and then gradually the cart position is moving to the origin. The stability analysis is fairly simple, because it is carried out using Lyapunov's approach. Furthermore, some computer simulations have been performed in order to test the effectiveness of the proposed controller.

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